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PROBLEMS OF STABILITY WITH RESPECT TO PART OF THE VARIABLES†

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The Lyapunov–Malkin theorem on stability and (simultaneously) exponential asymptotic stability with respect to part of the variables in the linear approximation in critical cases (in Lyapunov's sense) has served as a point of departure for various previous results. These results are strengthened by relaxing all additional assumptions (other than continuity) regarding the coefficients of the linear part of the non-linear system under consideration. The result is extended to the problem of polystability with respect to part of the variables. In addition, a method for narrowing down the admissible domain of variation of "uncontrollable" variables is worked out as applied to problems of asymptotic stability with respect to part of the variables. Examples are considered.
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To date, a large number of investigations have been devoted to the problem of stability with respect not to all variables characterizing the state of a system but only to a certain *given part* of those variables [1, 2]. The problem is often also referred to as that of *partial stability*. Some idea of the present state in this area may be obtained, for example, from [3–8], which also provide an extensive bibliography.

The problem of partial stability is closely connected with two intensively investigated stability problems: stability with respect to *two measures* [9–11] and with respect to *given state functions* [1, 12]. In addition, in the last decade problems of *polystability* (polystability with respect to part of the variables) have been singled out and the first studies have been published [13, 14]; in this context different groups of phase variables (or of part of the phase variables) possess different types of stability.

In what follows, the following problems will be considered in the context of the general problem of partial stability.

1. *The problem of exponential asymptotic* $\mathbf{y}$*-stability and (simultaneously) uniform stability, in Lyapunov's sense, of the unperturbed motion* $\mathbf{x} = (\mathbf{y}^T, \mathbf{z}^T)^T = \mathbf{0}$ *of a non-linear non-autonomous system of ordinary differential equations of perturbed motion*

$$\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}), \quad \mathbf{X}(t, \mathbf{0}) \equiv \mathbf{0} \tag{0.1}$$

under fairly general assumptions concerning the vector function $\mathbf{X}: R^{n+1} \to R^n$. In this connection, stronger versions will be established for several known results [3, 6, 8, 15, 16] based on the Lyapunov–Malkin theorem [1, 17] on stability in the linear approximation in critical cases in Lyapunov's sense. (As regards the vector notation used in system (0.1) and in other systems of differential equations to be considered later, we shall assume throughout that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are column vectors of appropriate dimensions; a superscript T will denote transposition.)

Together with the above problem, which may be treated as the problem of polystability of the unperturbed motion $\mathbf{x} = \mathbf{0}$ of system (0.1), we will also consider the more general problem of polystability with respect to part of the variables.

2. *The problem of asymptotic* $\mathbf{y}$*-stability of the unperturbed motion of system (0.1) without the additional assumption that the motion is stable in Lyapunov's sense.* A method proposed in [18] for narrowing down the domain in which the "uncontrollable" variables are allowed to vary is developed to study problems of asymptotic stability with respect to part of the variables. Previous results [3, 19, 20], going back to a theorem of Marachkov [21], are modified in this direction.

1. GENERALIZATION OF THE LYAPUNOV–MALKIN THEOREM

Stipulating, in keeping with the special features of partial stability problems, that the phase vector of the system is divided into two parts, we present system (0.1) as two groups of equations

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$$\dot{\mathbf{y}} = A(t)\mathbf{y} + B(t)\mathbf{z} + \mathbf{Y}(t,\mathbf{y},\mathbf{z}), \quad \dot{\mathbf{z}} = C(t)\mathbf{y} + D(t)\mathbf{z} + \mathbf{Z}(t,\mathbf{y},\mathbf{z}) \tag{1.1}$$

where A, B, C and D are matrix functions of t of appropriate dimensions, whose elements are functions continuous in $t \in [0, +\infty)$. The non-linear perturbations $\mathbf{Y}$ and $\mathbf{Z}$ are continuous and satisfy the conditions of the existence and uniqueness theorems in the domain $t \geqslant 0$, $\|\mathbf{x}\| \leqslant h = \text{const} > 0$.

Let $\mathbf{x}(t; t_0, \mathbf{x}_0)$ be a solution of system (1.1) satisfying the initial condition $\mathbf{x}_0 = \mathbf{x}(t_0; t_0, \mathbf{x}_0)$. The concept of polystability may be defined rigorously as follows.

Definition 1 [3, 22]. The unperturbed motion $\mathbf{y} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$ of system (1.1) is *uniformly stable in Lyapunov's sense and (simultaneously) exponentially asymptotically* $\mathbf{y}$*-stable* if, for any $\varepsilon > 0$, $t_0 \geqslant 0$, numbers $\delta(\varepsilon) > 0$ and $\gamma > 0$ exist such that, whenever $\|\mathbf{x}_0\| < \delta$, the following inequalities hold for all $t \geqslant t_0$

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| < \varepsilon, \quad \|\mathbf{y}(t;t_0,\mathbf{x}_0)\| < \varepsilon \exp[-\gamma(t-t_0)]$$

We assume that the following conditions are satisfied [3]

$$\mathbf{Y}(t,\mathbf{0},\mathbf{0}) \equiv \mathbf{Y}(t,\mathbf{0},\mathbf{z}) \equiv \mathbf{0}, \quad \mathbf{Z}(t,\mathbf{0},\mathbf{0}) \equiv \mathbf{Z}(t,\mathbf{0},\mathbf{z}) \equiv \mathbf{0}$$

$$\frac{\|\mathbf{Y}(t,\mathbf{y},\mathbf{z})\| + \|\mathbf{Z}(t,\mathbf{y},\mathbf{z})\|}{\|\mathbf{y}\|} \to 0 \quad \text{as} \quad \|\mathbf{y}\| + \|\mathbf{z}\| \to 0 \tag{1.2}$$

Theorem 1. Let the trivial solution of the linear system

$$\dot{\mathbf{y}} = A(t)\mathbf{y} + B(t)\mathbf{z}, \quad \dot{\mathbf{z}} = C(t)\mathbf{y} + D(t)\mathbf{z} \tag{1.3}$$

be uniformly stable in Lyapunov's sense and (simultaneously) exponentially asymptotically $\mathbf{y}$-stable. Then, if conditions (1.2) are satisfied, the unperturbed solution $\mathbf{y} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$ of the non-linear system (1.1) has the same stability property.

Proof. By the assumptions of the theorem, a Lyapunov V-function for the linear system (1.3) exists which satisfies the following conditions for all $t \geqslant 0$, $\|\mathbf{x}\| < \infty$ [22]

$$\|\mathbf{y}\| \leqslant V(t,\mathbf{x}) \leqslant M\|\mathbf{x}\|, \quad \dot{V}_{(1.3)}(t,\mathbf{x}) \leqslant -\alpha V(t,\mathbf{x}) \tag{1.4}$$

$$|V(t,\mathbf{x}'') - V(t,\mathbf{x}')| \leqslant M\|\mathbf{x}'' - \mathbf{x}'\| \quad (\alpha, M = \text{const} > 0) \tag{1.5}$$

The derivative of this V-function along trajectories of the non-linear system (1.1) may be represented as follows:

$$\dot{V}_{(1.1)}(t,\mathbf{x}) \leqslant -\alpha V(t,\mathbf{x}) + R(t,\mathbf{x}), \quad R = \left\langle \frac{\partial V}{\partial \mathbf{x}} \mathbf{X}^*(t,\mathbf{x}) \right\rangle, \quad \mathbf{X}^* = (\mathbf{Y}^T, \mathbf{Z}^T)^T$$

where $\langle \cdot \rangle$ denotes the scalar product.

By conditions (1.2) and (1.5), in the domain $t \geqslant 0$, $\|\mathbf{x}\| \leqslant h$ we have an estimate $|R(t, \mathbf{x})| \leqslant \varepsilon M\|\mathbf{y}\|$ which, in view of the first inequality in (1.4), can be written in the form

$$|R(t,\mathbf{x})| \leqslant \varepsilon MV \quad (\varepsilon = \text{const} \to 0 \quad \text{as} \quad \|\mathbf{x}\| \to 0)$$

Hence a number β $(0 < \beta < h)$ exists such that in the domain $t \geqslant 0$, $\|\mathbf{x}\| \leqslant \beta$

$$\dot{V}_{(1.1)}(t,\mathbf{x}) \leqslant -\alpha_1 V(t,\mathbf{x}) \quad (\alpha_1 = \text{const} > 0) \tag{1.6}$$

Consider an arbitrary solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of system (1.1) with initial data in the domain

$$t_0 \geqslant 0, \quad \|\mathbf{x}_0\| \leqslant \delta \quad (0 < \delta = \text{const} < \beta) \tag{1.7}$$

This solution satisfies the condition

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| \leqslant \beta \tag{1.8}$$

at least in some interval $T = (t_0, t^*)$. Therefore, by inequality (1.6) and the first group of inequalities (1.4), for $t \in T$, we have

$$\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| \leq V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leq M \| \mathbf{x}_0 \| \exp[-\alpha_1 (t - t_0)] \tag{1.9}$$

Conditions (1.2) and inequality (1.9) enable us to derive the estimate

$$\| \mathbf{X}(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \| \leq \alpha_2 \| \mathbf{x}_0 \| \exp[-\alpha_1 (t - t_0)] \tag{1.10}$$

$$t \in T, \quad \alpha_2 = \text{const} \to 0 \ \text{ as } \ \| \mathbf{x} \| \to 0$$

Using Cauchy's formula, we can express the solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of the non-linear system (1.1) as

$$\mathbf{x}(t; t_0, \mathbf{x}_0) = K(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} K(t, \tau)\mathbf{X}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_0)) d\tau \tag{1.11}$$

where $K(t, t_0) = U(t)U^{-1}(t_0)$ is the Cauchy matrix and $U(t)$ is a fundamental matrix of solutions of linear system (1.3).

Since the trivial solution of linear system (1.3) is uniformly stable in Lyapunov's sense, a number $N = \text{const} > 0$ exists such that $|K(t, t_0)| \leq N$ for $t \geq t_0$, $t_0 \geq 0$. It therefore follows from (1.11), on the basis of estimate (1.10), that

$$\| \mathbf{x}(t; t_0, \mathbf{x}_0) \| \leq N(1 + \alpha_1^{-1}\alpha_2) \| \mathbf{x}_0 \| \tag{1.12}$$

Let ε be an arbitrarily small number, $0 < \varepsilon < \beta$.

Choose $\delta(\varepsilon) > 0$ so small that $\delta < \min\{M^{-1}, [N(1 + \alpha_1^{-1}\alpha_2)]^{-1}\}\varepsilon$. Then it follows from (1.9) and (1.12) that for $t \in T$

$$\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| < \varepsilon \exp[-\alpha_1 (t - t_0)], \quad \| \mathbf{x}(t; t_0, \mathbf{x}_0) \| < \varepsilon \tag{1.13}$$

Thus, inequalities (1.13) hold throughout the time interval in which condition (1.8) holds. Since $\varepsilon < \beta$, it follows that inequalities (1.13) hold for all $t > t_0$. Consequently, the unperturbed solution $\mathbf{y} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$ of non-linear system (1.1) is uniformly stable in Lyapunov's sense and (simultaneously) exponentially asymptotically $\mathbf{y}$-stable.

Discussion of Theorem 1. 1. Theorem 1 extends certain results of [3, 6, 8, 15–17]. In [6, 8, 15, 17] the matrix functions A, B, C and D are independent of t; also in [17] additionally $B \equiv 0$, $D \equiv 0$ (all elements of the matrices B and D vanish identically), while in [15] $B \equiv 0$. In [3, 16] the matrix functions A, C and D depend on t, but $B \equiv 0$ and, in addition, all elements of the matrix functions A and C are bounded for $t \in [0, +\infty)$.

2. Confining themselves to the case $B \equiv 0$, Rumyantsev and Oziraner [3, 16] considered the more general class of **Z**-non-linearities, while in [6, 8] the more general class of **Y**-non-linearities is considered. In [6, 8], however, the matrix functions A, C and D do not depend on t.

3. The statement of the theorem is no longer true if the Lyapunov-stability of the trivial solution of linear system (1.3) is not uniform. This is demonstrated by an example due to Perron [17, 23], constructed for the case $B \equiv \mathbf{0}$, $C \equiv \mathbf{0}$.

Conditions (1.2) imposed on **Y** and **Z** may be somewhat weakened.

Corollary. Theorem 1 also holds if there are in system (1.1) additional non-linear terms $\mathbf{Y}^0$ and $\mathbf{Z}^0$ for which, in the domain $t \geq 0$, $\|\mathbf{x}\| \leq h$, the condition $|\mathbf{Y}^0(t, \mathbf{x})| + |\mathbf{Z}^0(t, \mathbf{x})| \leq \gamma V$ holds, where V is a Lyapunov function for linear system (1.3) which satisfies conditions (1.4) and (1.5), and γ is a sufficiently small positive constant.

Proof. The first part of the proof, including estimate (1.10), is as before. We further represent the solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of the non-linear system in terms of Cauchy's formula. Using estimates (1.9), (1.10) and the inequality $V(t, \mathbf{x}) \leq M\|\mathbf{x}\|$, we have

$$\| \mathbf{x}(t; t_0, \mathbf{x}_0) \| \leq N[(1 + \alpha_2\alpha_3^{-1}) \| \mathbf{x}_0 \| + \gamma M \int_{t_0}^{t} \| \mathbf{x}_0(\tau; t_0, \mathbf{x}_0) \| d\tau]$$

Applying the Gronwall–Bellman lemma [3] to this inequality, we obtain

$$\| \mathbf{x}(t; t_0, \mathbf{x}_0) \| \leq N(1 + \alpha_2\alpha_3^{-1}) \| \mathbf{x}_0 \| \exp(\gamma M)$$

The proof is now completed in the same way as that of Theorem 1.

Remarks. 1. Previously [3, 16], it was assumed that $B \equiv 0$, $\mathbf{Y}^0(t, \mathbf{x}) \equiv 0$, but the conditions considered for $\mathbf{Z}^0$ were more general.

2. If the coefficients of the matrix functions A, B and C, D are periodic functions, analytic for $t \in [0, +\infty)$, then Theorem 1 can be proved along lines similar to those of [6, 8]. At the same time, one can also slightly weaken the conditions imposed on the non-linear perturbations $\mathbf{Y}$ and $\mathbf{Z}$. However, the scheme of [6, 8] does not carry over to the general case, in which A, B, C and D depend on t.

Example 1. The equations of angular motion of a rigid body about its centre of mass under the action of linear torques are

$$\dot{\mathbf{x}} = L(t)\mathbf{x} + \mathbf{X}^*(\mathbf{x}), \quad \mathbf{x} = (y_1, y_2, z_1)^T$$

$$\mathbf{X}^* = [(J_2 - J_3)J_1^{-1}y_2z_1, \ (J_3 - J_1)J_2^{-1}y_1z_1, \ (J_1 - J_2)J_3^{1}y_1y_2]^T \tag{1.14}$$

where y_1, y_2, z_1 are the projections of the angular velocity vector $\mathbf{x}$ of the body onto the principal central axes of inertia, J_i are the principal central moments of inertia, and L is a 3×3 matrix whose elements are functions of $t \in [0, +\infty)$ characterizing the action of linear torques of dissipative and accelerating forces on the body.

Suppose the trivial solution $\mathbf{x} = (y_1, y_2, z_1)^T = \mathbf{0}$ of the linear system

$$\dot{\mathbf{x}} = L(t)\mathbf{x} \tag{1.15}$$

is uniformly stable in Lyapunov's sense and (simultaneously) exponentially asymptotically (y_1, y_2)-stable. The structure of the non-linear terms in system (1.14) is such that they satisfy conditions (1.2). Therefore, we conclude from Theorem 1 that the aforementioned stability property for linear system (1.15) also holds for the equilibrium position $\mathbf{x} = (y_1, y_2, z_1)^T = \mathbf{0}$ of non-linear system (1.14).

Note that system (1.14) does not satisfy all the conditions of the Lyapunov–Malkin Theorem as stipulated in [3, 6, 8, 15–17].

Example 2. Under fairly general assumptions, the motion of a holonomic mechanical system subject to linear and also non-linear forces of a general nature is described by a system of differential equations

$$\ddot{\mathbf{x}} = Q(t)\mathbf{x} + P(t)\dot{\mathbf{x}} + \mathbf{X}^*(t, \mathbf{x}, \dot{\mathbf{x}}), \quad \mathbf{x} = (\mathbf{y}^T, \mathbf{z}^T)^T \tag{1.16}$$

Let us assume that the matrix functions Q and P are continuous for $t \in [0, +\infty)$, and the non-linear vector function $\mathbf{X}^*$ is continuous in the domain $t \geqslant 0$; $\|\mathbf{x}\| + \|\dot{\mathbf{x}}\| = h$; in addition, system (1.16) satisfies the assumptions of the existence and uniqueness theorems.

Suppose the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of the linear system

$$\ddot{\mathbf{x}} = Q(t)\mathbf{x} + P(t)\dot{\mathbf{x}} \tag{1.17}$$

is uniformly stable in Lyapunov's sense and (simultaneously) exponentially asymptotically $(\mathbf{y}, \dot{\mathbf{y}})$-stable. Suppose, moreover, that the structure of the non-linear forces $\mathbf{X}^*$ is such that they satisfy the conditions

$$\mathbf{X}^*(t, \mathbf{0}, \mathbf{0}) \equiv \mathbf{0}, \quad \mathbf{X}^*(t, \mathbf{x}, \dot{\mathbf{x}}) \equiv \mathbf{0} \ \text{ for } \ \mathbf{y} = \dot{\mathbf{y}} = \mathbf{0}$$

$$\|\mathbf{X}^*(t, \mathbf{x}, \dot{\mathbf{x}})\| [\|\mathbf{y}\| + \|\dot{\mathbf{y}}\|]^{-1} \to 0 \ \text{ as } \ \|\mathbf{x}\| + \|\dot{\mathbf{x}}\| \to 0 \tag{1.18}$$

Then, by Theorem 1, the stability property specified for linear system (1.17) also holds for the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of non-linear system (1.16).

Suppose, in particular, that $\dim(\mathbf{x}) = 2$ and let system (1.16) have the form

$$\ddot{y}_1 = q_1y_1 + q_2\dot{y}_1 + e^{-\alpha t}z_1 + p_2\dot{z}_1 + Y_1(t, \mathbf{x}, \dot{\mathbf{x}}), \quad \ddot{z}_1 = p_1\dot{z}_1 + Z_1(t, \mathbf{x}, \dot{\mathbf{x}}) \tag{1.19}$$

where $q_i, p_i (i = 1, 2)$ and α are certain constants. If $q_i < 0, p_1 < 0, \alpha > 0$, then the equilibrium position $y_1 = \dot{y}_1 = z_1 = \dot{z}_1 = 0$ of the linear part of system (1.19) is uniformly Lyapunov-stable and (simultaneously) exponentially asymptotically $(y_1, \dot{y}_1)$-stable for any p_2. If the non-linear forces Y_1 and Z_1 satisfy conditions of type (1.18), then the same stability property will hold for the equilibrium position of the non-linear system (1.19) itself.

2. PARTIAL STABILIZATION OF THE STEADY MOTIONS OF A RIGID BODY

In applications, stabilization of the steady motions of a rigid body (such as a spacecraft) is frequently achieved by means of rotating masses attached to the body: flywheels and/or power gyroscopes.

In the stabilization process, these masses "take upon themselves" perturbations which occur as a result of the body's deviation from a given state [24–26].

We will show, however, that if the steady motions of the rigid body are stabilized only *partially* (that is, with respect to part of the variables), which is sufficient in many cases of practical importance, then the masses attached to the body may *only "transfer"* (without "taking upon themselves") perturbations to the part of the variables not controlled by the stabilization.

Suppose we have an asymmetric rigid body, with the axis of rotation of a uniform symmetric flywheel attached along one of the principal central axes of inertia of the body. The angular motion of the (gyrostat) system about its centre of mass is described by the equations [25]

$$(J_1 - A_1)\dot{x}_1 = (J_2 - J_3)x_2x_3 - u_1, \quad J_2\dot{x}_2 = (J_3 - J_1)x_1x_3 - A_1x_3\dot{\varphi}$$
$$J_3\dot{x}_3 = (J_1 - J_2)x_1x_2 + A_1x_2\dot{\varphi}, \quad A_1(\ddot{\varphi} + \dot{x}_1) = u_1 \tag{2.1}$$

where J_i are the principal central moments of inertia of the gyrostat, x_i are the projections of the angular velocity vector of the main body onto the principal central axes of inertia s_i of the gyrostat, A_1 and φ are the axial moment of inertia and angular velocity of the flywheel's own motion and u_1 is the controlling torque applied to the flywheel.

Equations (2.1) have the solution

$$x_1 = x_2 = 0, \quad x_3 = \omega = \text{const} > 0, \quad \dot{\varphi} = 0, \quad u_1 = 0 \tag{2.2}$$

corresponding to permanent rotation ("twist") of the main body of the gyrostat at a constant angular velocity ω about the s_3 axis. In this motion the flywheel, whose axis of rotation is attached along the s_1 axis, is fixed relative to the main body, while the direction of the vector $\mathbf{K}$ of angular momentum of the gyrostat coincides with the direction of the s_3 axis.

Introducing new variables $y_j = x_j$ $(j = 1, 2)$, $y_3 = \dot{\varphi}$, $z_1 = x_3 - \omega$, we set up a system of equations for the deviations from solution (2.2)

$$(J_1 - A_1)\dot{y}_1 = (J_2 - J_3)y_2(z_1 + \omega) - u_1, \quad J_2\dot{y}_2 = [(J_3 - J_1)y_1 - A_1y_3](z_1 + \omega)$$
$$(J_1 - A_1)\dot{y}_3 = (J_3 - J_2)y_2(z_1 + \omega) + J_1A_1^{-1}u_1 \tag{2.3}$$
$$J_3\dot{z}_1 = [(J_1 - J_2)y_1 + A_1y_3]y_2$$

Let us consider the problem of *partial* stabilization of the motion $\mathbf{y} = (y_1, y_2, y_3)^T = \mathbf{0}$, $z_1 = 0$ of system (2.3) y-stabilization by means of the control u_1. In this context, stabilization with respect to y_1, y_2 means that one must suppress small precessional and nutational oscillations of the angular momentum vector $\mathbf{K}$ of the gyrostat about the s_i axes attached to the body. Additional stabilization with respect to y_3 means that in the process of the (y_1, y_2)-stabilization, the flywheel only "*transfers*" the small perturbations to the "additional rotation" of the gyrostat about the s_3 axis of rotation.

Proposition. If $J_2 \neq J_3$, solution of the y-stabilization problem for unperturbed motion $\mathbf{y} = \mathbf{0}$, $z_1 = 0$ of system (2.3) yields the control law

$$u_1 = K\mathbf{y} \tag{2.4}$$

where K is some constant 1×3 row-vector.

Proof. Let us consider the linear subsystem describing the behaviour of the y-variables of the linear part of system (2.3). If $J_2 \neq J_3$, this subsystem is completely controllable [27]. Therefore the coefficients of the vector K in (2.4) may be chosen so that the trivial solution $\mathbf{y} = \mathbf{0}$, $z_1 = 0$ of the non-linear part of system (2.3) will be uniformly Lyapunov-stable and (simultaneously) exponentially y-stable.

The right-hand sides of system (2.3) vanish at $\mathbf{y} = \mathbf{0}$. Therefore, by Theorem 1, the stability property specified for the linear part of system (2.3) will also hold for the unperturbed motion $\mathbf{y} = \mathbf{0}$, $z_1 = 0$ of the non-linear system (2.3).

Remarks. 1. The problem just considered is of interest in spacecraft dynamics, where it is important to achieve "twist" of the craft about one of the principal central axes of inertia (as a rule—the *greatest* one) [24, 28].

2. In technical terms, implementation of control law (2.4) reduces to the following. As long as the gyrostat is performing the given motion (2.2), the flywheel is at rest (control drive switched off). In the event of small perturbations, special devices produce a control torque (2.4) and transmit it to the flywheel. As a result, the main body of the gyrostat returns in time to its original steady rotation, and the flywheel to its state of rest.

3. THE CONDITION OF POLYSTABILITY WITH RESPECT TO PART OF THE VARIABLES

Retaining the previously introduced notation, let us consider a non-linear system of equations of perturbed motion, more general than (1.1)

$$\dot{\mathbf{y}} = A(t)\mathbf{y} + B(t)\mathbf{z} + \mathbf{Y}(t,\mathbf{y},\mathbf{z},\mathbf{w}), \quad \dot{\mathbf{z}} = C(t)\mathbf{y} + D(t)\mathbf{z} + \mathbf{Z}(t,\mathbf{y},\mathbf{z},\mathbf{w}) \tag{3.1}$$

$$\dot{\mathbf{w}} = \mathbf{W}(t,\mathbf{y},\mathbf{z},\mathbf{w})$$

We will assume that in the domain

$$t \geqslant 0, \quad \|\mathbf{x}\| \leqslant h, \quad \|\mathbf{w}\| < \infty \tag{3.2}$$

where $\mathbf{x} = (\mathbf{y}^T, \mathbf{z}^T)^T$, the right-hand sides of system (3.1) are continuous and satisfy the conditions of the existence and uniqueness theorem. In addition, we will assume that the solutions of system (3.1) are $\mathbf{w}$-continuable [3, 29], that is, they are defined for all $t \geqslant 0$ for which $\|\mathbf{x}\| \leqslant h$.

The property we are studying—polystability with respect to part of the variables—may be defined rigorously as follows.

Definition 2. The unperturbed motion $\mathbf{y} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$, $\mathbf{w} = \mathbf{0}$ of system (3.1) is *uniformly* ($\mathbf{y}$, $\mathbf{z}$)-*stable and (simultaneously) exponentially asymptotically* $\mathbf{y}$-*stable*, if, for any $\varepsilon > 0$, $t_0 \geqslant 0$, numbers $\delta(\varepsilon) > 0$ and $\gamma > 0$ exist such that, whenever $\|\mathbf{x}_0\| + \|\mathbf{w}_0\| < \delta$, the following inequalities hold for all $t \geqslant t_0$

$$\|\mathbf{x}(t;\, t_0, \mathbf{x}_0, \mathbf{w}_0)\| < \varepsilon, \quad \|\mathbf{y}(t;\, t_0, \mathbf{x}_0, \mathbf{w}_0)\| < \varepsilon \exp[-\gamma(t - t_0)]$$

Suppose the following conditions hold in domain (3.2)

$$\begin{gathered} \mathbf{Y}(t,\mathbf{0},\mathbf{0},\mathbf{0}) \equiv \mathbf{Y}(t,\mathbf{0},\mathbf{z},\mathbf{w}) \equiv \mathbf{0} \\ \mathbf{Z}(t,\mathbf{0},\mathbf{0},\mathbf{0}) \equiv \mathbf{Z}(t,\mathbf{0},\mathbf{z},\mathbf{w}) \equiv \mathbf{0}, \quad \mathbf{W}(t,\mathbf{0},\mathbf{0},\mathbf{0}) \equiv \mathbf{0} \\ \frac{\|\mathbf{y}(t,\mathbf{y},\mathbf{z},\mathbf{w})\| + \|\mathbf{Z}(t,\mathbf{y},\mathbf{z},\mathbf{w})\|}{\|\mathbf{y}\|} \to 0 \ \text{ as } \ \|\mathbf{y}\| + \|\mathbf{z}\| \to 0 \end{gathered} \tag{3.3}$$

Theorem 2. Suppose the trivial solution of linear system (1.3) is uniformly stable in Lyapunov's sense and (simultaneously) exponentially asymptotically $\mathbf{y}$-stable. Then, if conditions (3.3) are satisfied, the unperturbed motion $\mathbf{y} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$, $\mathbf{w} = \mathbf{0}$ of non-linear system (3.1) is uniformly ($\mathbf{y}$, $\mathbf{z}$)-stable and (simultaneously) exponentially asymptotically $\mathbf{y}$-stable.

The proof follows the same lines as that of Theorem 1.

Discussion of Theorem 2. 1. Theorem 2 is an extension of the result of [15], where $B \equiv 0$ and the matrices A, C and D are constant.

2. The last condition of (3.3) is easily verified if it is known in advance that the solutions of system (3.1) are uniformly $\mathbf{w}$-bounded beginning in some sufficiently small neighbourhood of the unperturbed motion.

3. The conditions of Theorem 2 determine conditions for polystability with respect to part of the variables of the unperturbed motion of system (3.1): conditions of ($\mathbf{y}$, $\mathbf{z}$)-polystability.

4. A METHOD OF REDUCING THE ADMISSIBLE DOMAIN OF VARIATION OF "UNCONTROLLED" VARIABLES TO STUDY PARTIAL STABILITY PROBLEMS

Partial stability problems are frequently solved using the method of Lyapunov functions in a suitable modification. One such modification [18] reduces to adjusting the structure of the domain in which the Lyapunov functions are constructed. To elucidate: the domain usually considered in studying **y**-stability of the position $\mathbf{x} = (\mathbf{y}^T, \mathbf{z}^T)^T = \mathbf{0}$ of system (0.1)

$$t \geqslant 0, \quad \|\mathbf{y}\| \leqslant h, \quad \|\mathbf{z}\| < \infty \tag{4.1}$$

where h is a sufficiently small positive number, is *contracted*, being replaced by a domain

$$t \geqslant 0, \quad \|\mathbf{y}\| + \|\mathbf{W}(t, \mathbf{x})\| \leqslant h, \quad \|\mathbf{z}\| < \infty \tag{4.2}$$

where $\mathbf{W}(t, \mathbf{x})$ is some vector function, which depends on t and the phase variables of system (0.1). In this case, naturally, the new condition $\|\mathbf{y}\| + \|\mathbf{W}(t, \mathbf{x})\| \leqslant h$ must be verified while the problem is being solved.

The main point in studying the problem of **y**-stability in domain (4.2) is that the **y**-stable position $\mathbf{x} = (\mathbf{y}^T, \mathbf{z}^T)^T = \mathbf{0}$ of system (0.1) is always actually stable not only with respect to **y** but also with respect to certain functions $W_i = W_i(t, \mathbf{x})$. However, it is not always clear in advance just what W_i-functions are involved. In such a situation, suitable W_i-functions are naturally treated as an additional vector-valued Lyapunov **W**-function for the most rational substitute (4.2) for domain (4.1). When that is done it is not necessary to analyse the derivative of the **W**-function along trajectories of system (0.1), which is an added argument in favour of this approach.

Such an approach not only facilitates the construction of Lyapunov functions with appropriate properties, but also enables one to prove **y**-stability using functions which, even when $\dim(\mathbf{y}) = \dim(\mathbf{z}) = 1$, need not be of fixed sign [18] either with respect to **y** (in Rumyantsev's sense [2, 3]) or in Lyapunov's sense [2].

We will apply the above method of contracting the domain of variation of the "uncontrollable" variables to modify the conditions for asymptotic stability with respect to part of the variables [3].

We introduce the assumptions usually adopted in the theory of stability with respect to part of the variables [3, 29]: system (0.1) is continuous in domain (4.1), and its solution is unique and **z**-continuable. We will also consider two classes of functions: 1) functions $a_i(r)$: $R^1 \to R^1$ $(i = 1, 2, 3)$ which are continuous, monotone increasing for $r \in [0, h]$, and such that $a_i(0) = 0$; (2) a scalar function $V(t, \mathbf{x})$: $R^{n+1} \to R^1$, $n = \dim(\mathbf{x})$, $V(t, \mathbf{0}) \equiv 0$ and vector functions $\mathbf{W}(t, \mathbf{x})$: $R^{n+1} \to R^q$, $\mathbf{W}(t, \mathbf{0}) \equiv \mathbf{0}$, $\mathbf{U}(t, \mathbf{x})$: $R^{n+1} \to R^s$, $\mathbf{U}(t, \mathbf{0}) \equiv \mathbf{0}$; $q, s > 0$ are certain numbers whose designation depends on the specific problem being solved.

Theorem 3. Suppose a scalar f unction V and two vector functions **U** and **W** exist such that the following conditions hold in domain (4.2)

1. $V(t, \mathbf{x}) \geqslant a_1(\|\mathbf{y}\| + \|W(t, \mathbf{x})\|)$;
2. $\dot{V}(t, \mathbf{x}) \leqslant -a_2(\|\mathbf{U}(t, \mathbf{x})\|)$;
3. $\|\mathbf{U}(t, \mathbf{x})\| \geqslant a_3(\|\mathbf{y}\|)$;
4. a number $M(t_0, \mathbf{x}_0) > 0$ exists such that, for each of the functions U_i, either $\dot{U}_i \leqslant M$ or $\dot{U}_i \geqslant -M$.

Then the unperturbed motion $\mathbf{x} = \mathbf{0}$ of system (0.1) is asymptotically **y**-stable.

Proof. Conditions 1 and 2 imply that the unperturbed motion $\mathbf{x} = \mathbf{0}$ of system (0.1) is **y**-stable [2, 18]. In that case, for any $t_0 \geqslant 0$, $\Delta(t_0)$, $0 < \Delta < \delta$ exists, such that

$$\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| + \|\mathbf{W}(t; t_0, \mathbf{x}_0)\| < h, \; t \geqslant t_0 \text{ for } \|\mathbf{x}_0\| < \Delta$$

We will show that if $\|\mathbf{x}_0\| < \Delta$, then also

$$\lim \|\mathbf{U}(t, \mathbf{x}(t; t_0, \mathbf{x}_0))\| \to 0, \quad t \to \infty \tag{4.3}$$

Using the scheme of [3, 19–21], suppose the contrary: a number $l > 0$ and a vector $\mathbf{x}_*$ exist such that $\|\mathbf{x}_*\| < \Delta$, and a sequence $t_k \to \infty$, $t_k - t_{k-1} \geqslant \alpha > 0$ $(k = 1, 2, 3, \ldots)$ for which

$$\|\mathbf{U}(t_k, \mathbf{x}(t_k; t_0, \mathbf{x}_*))\| \geqslant l, \quad k = 1, 2, 3, \ldots \tag{4.4}$$

If Condition *4* is satisfied, one can find a number β $(0 < \beta < \alpha)$ such that, for $\|\mathbf{x}_*\| < \Delta$ and all $k = 1, 2, \ldots$

$$\tfrac{1}{2} l \leqslant \|\mathbf{U}(t, \mathbf{x}(t; t_0, \mathbf{x}_*))\| \leqslant h, \quad t \in T_k = [t_k - \beta, t_k + \beta] \tag{4.5}$$

Then it follows from (4.5) that for $\|x_*\| < \Delta$ the following inequalities hold along solutions $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_*)$

$$\dot{V}(t, \mathbf{x}) \leqslant -a_2(\|\mathbf{U}\|) \leqslant -a_2(\tfrac{1}{2}l), \quad t \in T_k$$

Therefore,

$$0 \leqslant V(t_k + \beta, \mathbf{x}(t_k + \beta; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) + \\ + \sum_{i=1}^{k} \int_{t_i - \beta}^{t_i + \beta} \dot{V}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_*)) d\tau \leqslant V(t_0, \mathbf{x}_0) - 2k\beta a_2\left(\frac{1}{2} l\right) \tag{4.6}$$

which is impossible for sufficiently large k.

Thus, our assumption (4.4) cannot be true. Consequently, $\lim a_3(\|\mathbf{y}(t; t_0, \mathbf{x}_0)\|) = 0$ as $t \to \infty$, provided that $\|\mathbf{x}_0\| < \Delta$. The theorem is proved.

Discussion of Theorem 3. 1. If $\mathbf{W} = \mathbf{0}$ Theorem 3 is a stronger version of Theorem 22.2 of [3], which extends the corresponding result of [20] to the case of partial asymptotic stability.

2. If $\mathbf{W} \neq \mathbf{0}$, then not only V and $\dot{V}$ but also $\|\mathbf{U}\|$ need not be sign-definite, either with respect to $\mathbf{y}$ (in Rumyantsev's sense [2, 3]) or in Lyapunov's sense [1]. In addition, Condition 4 may be verified in domain (4.2) but not in domain (4.1), and this extends the possibilities for using the theorem.

3. If $\mathbf{U} = \mathbf{y}$, Condition 4 reduces to the requirement that each component of the vector function $\mathbf{Y}$, defining the right-hand side of the first group of equations in system (0.1), should be bounded above or below. Therefore, when $\mathbf{U} = \mathbf{y}$, $\mathbf{W} = \mathbf{0}$, Theorem 3 reduces to a theorem of [3, 30] which extends the classical result of Marachkov [21] to the case of partial asymptotic stability.

4. The approach proposed here of using an additional Lyapunov vector function has also been used [31] to strengthen a number of theorems [3, 32] on asymptotic stability with respect to part of the variables for an autonomous system (0.1) (of the type of the Barbashin–Krasovskii theorem [33]).

Example 3. Let system (0.1) be

$$\dot{y}_1 = -y_1 + 2y_2 + \varepsilon' y_1 (z_1 z_2)^2 + y_1 z_1 z_2^2 z_3, \quad \dot{y}_2 = -2y_1 - y_2 - \varepsilon' y_2 (z_2 z_3)^2 \tag{4.7}$$
$$\dot{z}_1 = 2z_3 - 2\varepsilon' y_1^2 z_1, \quad \dot{z}_2 = \varepsilon' y_1^2 z_2, \quad \dot{z}_3 = -2z_1$$

Let us consider the problem of asymptotic (y_1, y_2)-stability of the unperturbed motion $y_1 = y_2 = z_i = 0$ ($i = 1, 2, 3$) of system (4.7). To do this, we introduce Lyapunov functions

$$V = y_1^2 + y_2^2 + (z_1 z_2)^2 + (z_2 z_3)^2, \quad \mathbf{W} = (W_1, W_2), \quad W_1 = z_1 z_2, \quad W_2 = z_2 z_3$$
$$\mathbf{U} = (U_1, U_2), \quad U_1 = y_1^2, \quad U_2 = y_2^2$$

Positive constants l, M_1 and M_2 exist such that the following relations hold in domain (4.2)

$$V = y_1^2 + y_2^2 + W_1^2 + W_2^2 \geqslant a_1(\|\mathbf{y}\| + \|\mathbf{W}\|)$$
$$\dot{V} = -2\left[\left(y_1^2 + y_2^2\right) - y_1^2 W_1 W_2\right] \leqslant -l\left(y_1^2 + y_2^2\right) \leqslant -a_2(\|\mathbf{U}\|), \quad \|\mathbf{U}\| \geqslant a_3(\|\mathbf{y}\|)$$
$$-M_1 \leqslant \dot{U}_1 = 2y_1\left\{-y_1 + 2y_2 + \varepsilon' y_1 W_1^2 + y_1 W_1 W_2\right\}, \qquad \dot{U}_2 = 2y_2\left\{-2y_1 - y_2 - \varepsilon' y_2 W_2^2\right\} \leqslant M_2$$

Consequently, the functions V, $\mathbf{W}$ and $\mathbf{U}$ satisfy all the conditions of Theorem 3. Hence the equilibrium position $y_1 = y_2 = z_i = 0$ ($i = 1, 2, 3$) of system (4.7) is asymptotically (y_1, y_2)-stable.

Note that the relation $\dot{V} \leqslant -l(y_1^2 + y_2^2)$ is not guaranteed in domain (4.1), that is, the function $\dot{V}$ need not be y-sign-definite in Rumyantsev's sense [2, 3].

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